Dirac fields in the background of a magnetic flux string and spectral boundary conditions

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We study the problem of a Dirac field in the background of an Aharonov-Bohm flux string. We exclude the origin by imposing spectral boundary conditions at a finite radius then shrinked to zero. Thus, we obtain a behaviour of the eigenfunctions which is compatible with the self-adjointness of the radial Hamiltonian and the invariance under integer translations of the reduced flux.

After confining the theory to a finite region, we check the consistency with the index theorem, and evaluate its vacuum fermionic number and Casimir energy.

I. INTRODUCTION

The consideration of Bohm-Aharonov [1] scenarios is relevant to different physical problems, mainly 2+1-dimensional models in superconductivity and particle theory. The inclusion of the spin is an aspect to which much attention has been devoted. Many authors have studied this point in connection with the interaction of cosmic strings with matter [2–6].

We will treat the $3+1$-dimensional problem of a Dirac field in the presence of a flux string. Due to the symmetry of the background field, the original four dimensional Dirac Hamiltonian can be written in a block-diagonal form, thus leading to two $2+1$-dimensional problems.

In this context, after developing the eigenfunctions in a convenient angular basis, the need to consider self-adjoint extensions of the radial Dirac Hamiltonian was recognized [7,8]. A one-parameter family of boundary conditions at the origin was then shown to arise, which amounts to establishing a relationship between components of the spinor rather than asking for the simultaneous finiteness of both of them. However, it was shown in [9] that only two values of the extension parameter correspond to the presence of a Dirac delta magnetic field at the origin.

In references [2,5,6], one of this possible boundary conditions was obtained, starting from a model in which the continuity of both components of the Dirac spinor is imposed at finite radius and the zero-radius limit is taken. As pointed out in [2,10–13],when this boundary condition is imposed at the origin, the invariance under integer translations of the reduced magnetic flux (or, equivalently, large gauge transformations) is lost.

In this paper, we adopt the view that the origin is an excluded point. The plane is thus a punctured one, which has the topology of a cylinder, and the aforementioned singular gauge transformations must constitute an invariance of the theory. It is in this spirit that, after devoting Sec. II to setting the problem, in Sec. III we impose on the Dirac fields spectral boundary conditions of the Atiyah- Patodi- Singer (APS) type [14–17], taken as in reference [18,19], at a finite radius. We show that, when the zero radius limit is taken one then obtains, at the origin, other among the boundary conditions compatible with self-adjointness, which present a minimal divergence behaviour [11,12,10]. They respect the above mentioned periodicity of the flux and are, at the same time, compatible with the presence of a Dirac delta magnetic field at the origin.

From Sec. IV on, we confine the whole system within a punctured circle of radius R, imposing at the external boundary spectral conditions complementary to those satisfied at the position of the flux tube. We discuss the problem of zero modes and the compatibility with APS index theorem for manifolds with boundaries.

In Sec. V, we determine the energy spectrum and study the fermionic number and Casimir energy of the confined system. This last calculation is performed in a ζ function approach, following the lines of reference [20].

Finally, Sec. VI contains some comments and conclusions.

II. SETTING OF THE PROBLEM

We study the Dirac equation for a massless particle in four dimensional Minkowski space.

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$$(i\not{\partial} - \not{A})\,\Psi = 0 \tag{1}$$

in the presence of a flux tube located at the origin, i.e.,

$$\vec{H} = \vec{\nabla} \wedge \vec{A} = \frac{\kappa}{r}\delta(r)\breve{e}_z \tag{2}$$

where $\kappa = \frac{\Phi}{2\pi}$ is the reduced flux.

As the gauge potential is $z-$independent, equation (1) can be decoupled into two uncoupled two-component equations [5],by choosing:

$$\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \tag{3}$$

In order to avoid singularities, we will consider that only $r > r_0$ is accessible, and take the limit $r_0 \to 0$,which is equivalent to having a punctured plane with the removed point corresponding to the string position.

By taking

$$A_z = 0 \qquad A_r = 0 \qquad A_\theta = \frac{\kappa}{r} \qquad ,\text{for} \quad r > r_0 \tag{4}$$

the Hamiltonian can be seen to be block-diagonal

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{5}$$

where the two-by-two blocks are given by

$$H_\pm = \begin{pmatrix} 0 & ie^{\mp i\theta}\left(\partial_r \pm \mathrm{B}\right) \\ -ie^{\pm i\theta}\left(-\partial_r \pm \mathrm{B}\right) & 0 \end{pmatrix} \tag{6}$$

with

$$\mathrm{B} = -\frac{i}{r}\partial_\theta - \frac{\kappa}{r} \tag{7}$$

It should be noticed that these two "polarizations", which we will label with $s = \pm 1$, correspond to the two inequivalent choices for the gamma matrices in 2+1 dimensions [6].

From now on, we will be working with $s = 1$ (the case $s = -1$ can be studied in a similar way, and explicit reference will be made to it whenever necessary).

In this case, we can write:

$$H_+ = \begin{pmatrix} 0 & L^\dagger \\ L & 0 \end{pmatrix} \qquad ,\text{with} \quad L = -ie^{i\theta}\left(-\partial_r + \mathrm{B}\right) \qquad L^\dagger = ie^{-i\theta}\left(\partial_r + \mathrm{B}\right) \tag{8}$$

and its eigenfunctions:

$$\Psi_E = \begin{pmatrix} \varphi_E(r,\theta) \\ \chi_E(r,\theta) \end{pmatrix} \qquad , \quad \text{satisfy}: \quad \begin{matrix} L\varphi_E = E\chi_E \\ L^\dagger\chi_E = E\varphi_E \end{matrix} \tag{9}$$

Now, the two components in Ψ_E have different θ dependence. In order to make this fact explicit, and to discuss boundary conditions at $r = r_0$, we introduce [18,19]

$$\Psi_E = \frac{1}{\sqrt{r}}\begin{pmatrix} e^{-i\frac{\theta}{2}}\varphi_{1E}(r,\theta) \\ e^{i\frac{\theta}{2}}\chi_{1E}(r,\theta) \end{pmatrix} \tag{10}$$

and

$$L_1 = -\partial_r + \mathrm{B} \qquad L_1^\dagger = \partial_r + \mathrm{B} \qquad , \tag{11}$$

so that

$$L = -i\frac{e^{i\frac{\theta}{2}}}{\sqrt{r}}L_1 e^{i\frac{\theta}{2}}\sqrt{r} \qquad L^{\dagger} = i\frac{e^{-i\frac{\theta}{2}}}{\sqrt{r}}L_1^{\dagger} e^{-i\frac{\theta}{2}}\sqrt{r} \tag{12}$$

and

$$\begin{aligned} L_1 \varphi_{1E} &= iE\chi_{1E} \\ L_1^{\dagger}\chi_{1E} &= -iE\varphi_{1E} \end{aligned} \tag{13}$$

We expand φ_{1E} and χ_{1E} in terms of eigenfunctions of B

$$\mathrm{B}\, e_n = \lambda_n e_n \tag{14}$$

which are of the form:

$$e_n = e^{i\left(n+\frac{1}{2}\right)\theta} \qquad , \quad \text{with} \qquad \lambda_n\left(r\right) = \frac{n+\frac{1}{2}-\kappa}{r} \quad , \quad n\,\epsilon\, Z \tag{15}$$

once the condition has been imposed that φ_E and χ_E in equation (9) are single-valued in θ.

Thus,we have

$$\begin{aligned} \varphi_{1E}\left(r,\theta\right) &= \textstyle\sum_{n=-\infty}^{\infty} f_n\left(r\right) e^{i\left(n+\frac{1}{2}\right)\theta} \\ \chi_{1E}\left(r,\theta\right) &= \textstyle\sum_{n=-\infty}^{\infty} g_n\left(r\right) e^{i\left(n+\frac{1}{2}\right)\theta} \end{aligned} \tag{16}$$

Replacing in (13) we obtain, for noninteger κ ($\kappa = k+\alpha$, with k the integer part of κ and α its fractionary part)

$$\begin{aligned} \varphi_{1E}\left(r,\theta\right) &= \sqrt{r}\textstyle\sum_{n=-\infty}^{\infty}\left(A_n\, J_{n-\kappa}\left(|E|r\right) + B_n\, J_{\kappa-n}\left(|E|r\right)\right) e^{i\left(n+\frac{1}{2}\right)\theta} \\ \chi_{1E}\left(r,\theta\right) &= -i\frac{|E|}{E}\sqrt{r}\textstyle\sum_{n=-\infty}^{\infty}\left(A_n\, J_{n+1-\kappa}\left(|E|r\right) - B_n\, J_{\kappa-n-1}\left(|E|r\right)\right) e^{i\left(n+\frac{1}{2}\right)\theta} \end{aligned} \tag{17}$$

and

$$\Psi_E\left(r,\theta\right) = \begin{pmatrix} \sum_{n=-\infty}^{\infty}\left(A_n\, J_{n-\kappa}\left(|E|r\right) + B_n\, J_{\kappa-n}\left(|E|r\right)\right) e^{in\theta} \\ \sum_{n=-\infty}^{\infty} -i\frac{|E|}{E}\left(A_n\, J_{n+1-\kappa}\left(|E|r\right) - B_n\, J_{\kappa-n-1}\left(|E|r\right)\right) e^{i(n+1)\theta} \end{pmatrix} \tag{18}$$

(Of course, for integer κ, a linear combination of Bessel and Neumann functions must be taken).

Finally, for $s=-1$, the upper and lower components of Ψ_E interchange, and $E \to -E$.

III. BOUNDARY CONDITIONS AT THE ORIGIN

As is well known [2,7,8], the radial Dirac Hamiltonian in the background of an Aharonov-Bohm gauge field requires a self-adjoint extension for the critical subspace $n = k$. In fact, imposing regularity of both components of the Dirac field at the origin is too strong a requirement, except for integer flux. Rather, one has to apply the theory of Von Neumann deficiency indices [21], which leads to a one parameter family of allowed boundary conditions, characterized by [8]

$$i\lim_{r\to 0}\left(Mr\right)^{\nu+1} g_n(r)\sin\left(\frac{\pi}{4}+\frac{\Theta}{2}\right) = \lim_{r\to 0}\left(Mr\right)^{-\nu} f_n(r)\cos\left(\frac{\pi}{4}+\frac{\Theta}{2}\right) \tag{19}$$

with ν varying between -1 and 0 ($\nu = -\alpha$ for $s=1$; $\nu = \alpha - 1$ for $s=-1$) . Here, Θ parametrizes the admissible self-adjoint extensions, and M, a mass parameter, is introduced for dimensional reasons.

Which of these boundary conditions to impose depends on the physical situation under study.

One possibility is to take a finite flux tube, ask for continuity of both components of the Dirac field at finite radius and then let this radius go to zero [2,5,6]. Thus, one of the possible self-adjoint extensions is obtained, which corresponds to $\Theta = \frac{\pi}{2}\mathrm{sgn}\left(\kappa\right)$.

As pointed out in [2,10,12], this kind of procedure leads to a boundary condition that breaks the invariance under $\kappa \to \kappa + n$ ($n\,\epsilon\, Z$). Now, this is a large gauge symmetry, which of course is singular when considering the whole plane, but is not so when the origin is removed or, equivalently, the plane has the topology of a cylinder.

In order to preserve the aforementioned symmetry we propose, instead, to exclude the origin, by imposing spectral boundary conditions of the Atiyah-Patodi-Singer type [14–16], as defined in [18], at a finite radius r_0 and letting $r_0 \to 0$.

We consider the development in eq.(16) and impose, at $r = r_0$:

$$\begin{array}{lll} f_n\left(r_0\right) = 0 & \text{for} & \lambda_n\left(r_0\right) \leq 0 \\ g_n\left(r_0\right) = 0 & \text{for} & \lambda_n\left(r_0\right) > 0 \end{array} \tag{20}$$

for $s = 1$.

As is well known, imposing this kind of boundary condition is equivalent to removing the boundary, by attaching a semi-infinite tube at its position and then extending the Dirac equation by a constant extension of the gauge field, while asking that zero modes be square integrable [19,22] (except for $\lambda_n = 0$, where a constant zero mode remains, with a nonzero lower component).

Then, going to eq.(17) we have, after using the dominant behaviour of Bessel functions for small arguments :

$$\begin{array}{ll} \dfrac{A_n}{B_n} \sim r_0^{2(k+\alpha-n)} & n - k + \dfrac{1}{2} - \alpha \leq 0 \\ \dfrac{B_n}{A_n} \sim r_0^{2(n+1-k-\alpha)} & n - k + \dfrac{1}{2} - \alpha > 0 \end{array} \tag{21}$$

Now, we analyze two different situations:
If $\alpha \geq \frac{1}{2}$

$$\begin{array}{lll} \dfrac{A_n}{B_n} \to_{r_0 \to 0} 0 & \text{for} & n \leq k \\ \dfrac{B_n}{A_n} \to_{r_0 \to 0} 0 & \text{for} & n \geq k+1 \end{array} \tag{22}$$

and the eigenfunctions in eq(18) are of the form:

$$\Psi_E\left(r,\theta\right) = \sum_{n=-\infty}^{k} B_n \begin{pmatrix} J_{k+\alpha-n}\left(|E| r\right) e^{in\theta} \\ -i\frac{|E|}{E} J_{k+\alpha-n-1}\left(|E| r\right) e^{i(n+1)\theta} \end{pmatrix} + \sum_{n=k+1}^{\infty} A_n \begin{pmatrix} J_{n-k-\alpha}\left(|E| r\right) e^{in\theta} \\ i\frac{|E|}{E} J_{n+1-k-\alpha}\left(|E| r\right) e^{i(n+1)\theta} \end{pmatrix} \tag{23}$$

If $\alpha < \frac{1}{2}$

$$\begin{array}{lll} \dfrac{A_n}{B_n} \to_{r_0 \to 0} 0 & \text{for} & n \leq k-1 \\ \dfrac{B_n}{A_n} \to_{r_0 \to 0} 0 & \text{for} & n \geq k \end{array} \tag{24}$$

and the eigenfunctions are:

$$\Psi_E\left(r,\theta\right) = \sum_{n=-\infty}^{k-1} B_n \begin{pmatrix} J_{k+\alpha-n}\left(|E| r\right) e^{in\theta} \\ -i\frac{|E|}{E} J_{k+\alpha-n-1}\left(|E| r\right) e^{i(n+1)\theta} \end{pmatrix} + \sum_{n=k}^{\infty} A_n \begin{pmatrix} J_{n-k-\alpha}\left(|E| r\right) e^{in\theta} \\ i\frac{|E|}{E} J_{n+1-k-\alpha}\left(|E| r\right) e^{i(n+1)\theta} \end{pmatrix} \tag{25}$$

Notice that our procedure leads precisely to a self adjoint extension satisfying the condition of minimal irregularity (the radial functions diverge at $r \to 0$ at most as r^{-p}, with $p \leq \frac{1}{2}$). It corresponds to the values of the parameter Θ :

$$\Theta = \begin{cases} -\frac{\pi}{2} & \text{for}\, \alpha \geq \frac{1}{2} \\ \frac{\pi}{2} & \text{for}\, \alpha < \frac{1}{2} \end{cases} \tag{26}$$

As shown in [9] $\Theta = \pm\frac{\pi}{2}$ are the only two possible values of the parameter which correspond to having a Dirac delta magnetic field at the origin.

Moreover, this extension is compatible with periodicity in κ. In fact, the dependence on k can be reduced to an overall phase factor in the eigenfunctions.

As regards charge conjugation

$$\Psi_E \to \sigma_1 {\Psi_E}^* \quad ; \quad \kappa \to -\kappa$$

it is respected by the eigenfunctions, except for $\alpha = \frac{1}{2}$. This is due to the already commented presence of a constant zero mode on the cylinder.

Notice that, for the representation $s = -1$ of 2×2 Dirac matrices,

$$\Psi_E^{(-)} = \begin{pmatrix} e^{i\frac{\theta}{2}} \sum f_n e^{i\left(n+\frac{1}{2}\right)\theta} \\ e^{-i\frac{\theta}{2}} \sum g_n e^{i\left(n+\frac{1}{2}\right)\theta} \end{pmatrix} \tag{27}$$

and APS boundary conditions must be reversed, for $\lambda \neq 0$, as compared to (20), since the operator B changes into $-$B. For $\lambda = 0$ the lower component will be taken to be zero at r_0 which, as we will show later, allows for charge conjugation to be a symmetry of the whole model. Thus we take

$$\begin{array}{ll} f_n(r_0) = 0 & \text{for} \quad \lambda_n(r_0) > 0 \\ g_n(r_0) = 0 & \text{for} \quad \lambda_n(r_0) \leq 0 \end{array} \tag{28}$$

for $s = -1$

In this case, the resulting extension corresponds to

$$\Theta = \begin{cases} \frac{\pi}{2} & \text{for}\, \alpha \geq \frac{1}{2} \\ -\frac{\pi}{2} & \text{for}\, \alpha < \frac{1}{2} \end{cases} \tag{29}$$

This is an example of a physical application of APS boundary conditions, which are generally chosen due to their mathematical interest. We wish to stress that our conclusions concerning the behaviour at the origin hold true for massive Dirac fields. The procedure of imposing boundary conditions at a finite radius, then taken to zero, was considered for massive fields in [23], where some vacuum quantum numbers were examined.

It is worth pointing that, for integer $\kappa = k$, our procedure leads (both for $s = \pm 1$) to the requirement of regularity of both components at the origin. In this case

$$\Psi_E(r,\theta) = \sum_{n=-\infty}^{\infty} A_n \begin{pmatrix} J_{n-\kappa}(|E|r)\, e^{in\theta} \\ i\frac{|E|}{E} J_{n+1-\kappa}(|E|r)\, e^{i(n+1)\theta} \end{pmatrix} \tag{30}$$

IV. THE THEORY IN A BOUNDED REGION. ZERO MODES AND APS INDEX THEOREM

From now on, we will confine the Dirac fields inside a bounded region, by introducing a boundary at $r = R$, and imposing there boundary conditions of the APS type, complementary to the ones considered at $r = r_0$.

For $s = 1$, these boundary conditions are

$$\begin{array}{ll} f_n(R) = 0 & \text{for} \quad \lambda_n(R) > 0 \\ g_n(R) = 0 & \text{for} \quad \lambda_n(R) \leq 0 \end{array} \tag{31}$$

We start by studying the zero modes of our theory, which are of the form

$$\Psi_0(r,\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} \sum_{n=-\infty}^{\infty} A_n r^{n-k-\alpha} e^{i\left(n+\frac{1}{2}\right)\theta} \\ e^{i\frac{\theta}{2}} \sum_{n=-\infty}^{\infty} B_n r^{k+\alpha-n-1} e^{i\left(n+\frac{1}{2}\right)\theta} \end{pmatrix} \tag{32}$$

The conditions at $r = r_0$ imply

$$\begin{array}{lll} A_n = 0 & , & \text{for} \quad n \leq k+\alpha-\frac{1}{2} \\ B_n = 0 & , & \text{for} \quad n > k+\alpha-\frac{1}{2} \end{array} \tag{33}$$

Now, the boundary conditions at R imply

$$\begin{array}{lll} A_n = 0 & , & \text{for} \quad n > k+\alpha-\frac{1}{2} \\ B_n = 0 & , & \text{for} \quad n \leq k+\alpha-\frac{1}{2} \end{array} \tag{34}$$

Thus, no zero mode remains under this boundary condition, even without taking $r_0 \to 0$. This is in agreement with the APS index theorem [14–16]. In fact, according to such theorem

$$n_+ - n_- = \mathcal{A} + b(r_0) + b(R) \tag{35}$$

where $n_+(n_-)$ is the number of chirality positive (negative) zero energy solutions, $\mathcal{A}$ is the anomaly, or bulk contribution, and b are the surface contributions coming from both boundaries [17,18]

$$b\left(R\right)=\frac{1}{2}\left(h_R-\eta\left(R\right)\right)\qquad b\left(r_0\right)=\frac{1}{2}\left(\eta\left(r_0\right)-h_{r_0}\right) \tag{36}$$

with

$$\eta\left(r\right)=\left.\sum_{\lambda_n(r)\neq 0}\mathrm{sgn}\lambda_n\left(r\right)\,\left|\lambda_n\left(r\right)\right|^{-s}\right]_{s=0} \tag{37}$$

the spectral asymmetry of the boundary operator B and h_r is the dimension of its kernel.

In our case :

$$b\left(r_0\right)=\begin{cases}\alpha-1 & \alpha>\frac{1}{2}\\ -\frac{1}{2} & \alpha=\frac{1}{2}\\ \alpha & \alpha<\frac{1}{2}\end{cases}\qquad b\left(R\right)=\begin{cases}1-\alpha & \alpha>\frac{1}{2}\\ \frac{1}{2} & \alpha=\frac{1}{2}\\ -\alpha & \alpha<\frac{1}{2}\end{cases} \tag{38}$$

Thus, the boundary contributions cancel. As regards the volume part, it also vanishes for the gauge field configuration under study, and we have $n_+-n_-=0$, which is consistent with our analysis following eq.(34).

For $s=-1$ both boundary contributions interchange, and identical conclusions hold regarding the index.

V. ENERGY SPECTRUM. FERMIONIC NUMBER AND CASIMIR ENERGY

The energy spectrum can be determined by imposing (for $s=1$) the boundary conditions (31) at $r=R$ on the eigenfunctions of eq.(23) if $\alpha\geq\frac{1}{2}$, or on the eigenfunctions in eq.(25), if $\alpha<\frac{1}{2}$. When doing so, we obtain :

$$E_{n,l}=\begin{cases}\pm\frac{j_{n-\alpha,l}}{R} & ,\quad n=1,\ldots\infty\quad l=1\ldots\infty\\ \pm\frac{j_{n+\alpha,l}}{R} & ,\quad n=-1,\ldots\infty\quad l=1\ldots\infty\end{cases}\qquad \text{for}\,\alpha\geq\frac{1}{2} \tag{39}$$

and

$$E_{n,l}=\begin{cases}\pm\frac{j_{n-\alpha,l}}{R} & ,\quad n=0,\ldots\infty\quad l=1\ldots\infty\\ \pm\frac{j_{n+\alpha,l}}{R} & ,\quad n=0,\ldots\infty\quad l=1\ldots\infty\end{cases}\qquad \text{for}\,\alpha<\frac{1}{2} \tag{40}$$

where $j_{\nu,l}$ is the l-th positive root of J_ν. The same spectrum results for $s=-1$.

For both s values, the energy spectrum is symmetric with respect to zero. This fact, together with the absence of zero modes results in a null vacuum expectation value for the fermionic charge [24] :

$$<N>_+=-\frac{1}{2}\left(n_+-n_-\right)=0 \tag{41}$$

For the same reasons $<N>_-=0$, so the total fermionic number of the theory is null.

It is interesting to note that the origin contributes to the fermionic number with (see the discussion in [23])

$$<N>_{r_0,\pm}=\begin{cases}\mp\frac{1}{2}\left(\alpha-1\right) & \alpha>\frac{1}{2}\\ \pm\frac{1}{4} & \alpha=\frac{1}{2}\\ \mp\frac{1}{2}\alpha & \alpha<\frac{1}{2}\end{cases} \tag{42}$$

which coincides, for each s value, with the result presented for the whole punctured plane in [10,12], where opposite signs of the mass correspond to our opposite signs of s (see also [26] for related work) , except for $\alpha=\frac{1}{2}$. This last fact is associated with charge conjugation non invariance in each subspace. However, the sum of both contributions cancels for all α.

We go now to the evaluation of the Casimir energy, which is formally given by

$$E_C=-\frac{1}{2}\left(\sum_{E>0}E_{n,l}-\sum_{E<0}E_{n,l}\right)=-\left(\sum_{E>0}E_{n,l}\right)\qquad , \tag{43}$$

where the symmetry of the energy spectrum for our problem has been used. Of course, a regularization method must be introduced in order to give sense to this divergent sum. In the framework of the ζ-regularization [27,28],

$$E_C = -\mu \sum_{E>0} \left(\frac{E_{n,l}}{\mu} \right)^{-z} \Bigg\rfloor_{z=-1} \tag{44}$$

where the parameter μ was introduced for dimensional reasons.

Here, it is useful to define the so-called partial zeta function as in references [20,29,30]

$$\zeta_\nu (z) = \sum_{l=1}^{\infty} (j_{\nu,l})^{-z} \tag{45}$$

So, for the problem at hand we have, from eqs.(39),(40),(44) and (45)

$$E_C = -2\mu \, (\mu R)^z \sum_\nu \zeta_\nu (z) \Bigg\rfloor_{z=-1} \tag{46}$$

with

$$\sum_\nu \zeta_\nu = \begin{cases} \sum_{n=-\infty}^{\infty} \zeta_{|n-\alpha|} + \zeta_{\alpha-1} & \text{for} \quad \alpha \geq \frac{1}{2} \\ \sum_{n=-\infty}^{\infty} \zeta_{|n-\alpha|} + \zeta_{-\alpha} & \text{for} \quad \alpha < \frac{1}{2} \end{cases} \tag{47}$$

Notice that, as a consequence of the invariance properties of the imposed boundary conditions, the Casimir energy is both periodic in κ and invariant under $\alpha \to 1-\alpha$, as well as continuous at integer values of κ.

For any value of κ, it is the sum of the energy corresponding to a scalar field in the presence of a flux string and subject to Dirichlet boundary conditions (studied in ref. [20]), plus a partial zeta coming, for fractionary κ, from the presence of an eigenfunction which is singular at the origin or, for integer κ, from the duplication of J_0.

Both contributions can be studied following the methods employed in [20] and developed by the same authors in previous work [29,30]. So, here we won't go into the details of such calculation.

The scalar field contribution presents a pole at $z = -1$, with an α-independent residue. So, in this case, a renormalized Casimir energy can be defined as

$$E_C^{ren} = E_C(\alpha) - E_C(0) \tag{48}$$

Following [20], this quantity can be obtained through a numerical calculation. (Notice there is a minor error in eq.(3.6) of that reference. In fact, a factor of π is missing in the fifth term on the right hand side of such equation).

Now, going back to the Dirac field, the partial zeta appearing in (46),(47) can also be evaluated with the methods of [20]. However, it presents a pole, whose residue is α-dependent :

$$\zeta_\nu (z) \rfloor_{z=-1} = F.P. + \frac{1}{8\pi} \left(1 - 4\nu^2\right) \frac{1}{z+1} \Bigg\rfloor_{z=-1} \tag{49}$$

Notice that such residue vanishes only for $\nu = \pm\frac{1}{2}$ ($\alpha = \frac{1}{2}$ in the case of interest).

The finite part of the Casimir energy is plotted in fig.(1) as a function of α for $\mu R = 1$. Of course, an absolute meaning cannot be assigned to it, due to the presence of the pole and the consequent need to introduce α-dependent counterterms.

VI. COMMENTS AND CONCLUSIONS

In summary, we have studied the problem of Dirac fields in the presence of a Bohm-Aharonov background which can, by virtue of the symmetry properties of the external field, be separated into two 2+1-dimensional ones (labelled through the paper by $s = \pm 1$).

We have adopted the viewpoint that the plane is a punctured one, so that the model must turn out to be invariant under integer translations of the reduced flux, which are nothing but "singular" gauge transformations. In this spirit we have shown that, by imposing spectral boundary conditions of the APS type at a finite radius and then letting this radius to zero, one among the family of allowed boundary conditions at the origin is obtained for the two-component spinors. Under this boundary condition, the dependence on the integer part of the flux can be extracted from the eigenfunctions of the Hamiltonian as an overall phase. As a consequence, the aforementioned symmetry is preserved.

As regards the invariance under charge conjugation, it is broken for each s value by the boundary conditions at the origin, for the fractionary part of the flux equal to $\frac{1}{2}$. This is due to the fact that a constant zero mode of definite handedness is allowed for on the cylinder when applying the APS boundary conditions. However, this symmetry is preserved when opposite handed conditions are imposed on both s-subspaces.

It is worth stressing that this approach to the problem of self-adjointness at the origin is, to our knowledge, the first proposal of a physical application of APS boundary conditions in this context.

We have then confined the theory to a finite region, i.e. a punctured circle, and imposed adequate APS boundary conditions at the exterior boundary, with the aim of checking consistency with the APS index theorem for manifolds with boundaries, which we have shown to hold.

Finally, we have studied the vacuum fermionic number and analized, through zeta function regularization techniques, the Casimir energy of the model. Since this last turns out to present a divergence depending on the fractionary part of the flux, no absolute significance can be assigned to its finite part, which is thus subject to renormalizations.

However, notice that, contrary to the treatment of the origin, APS boundary conditions were imposed at the exterior boundary for merely formal reasons (the existence of an index theorem for this case). The vacuum energy under local (bag-like) external boundary conditions is at present under study.

ACKNOWLEDGMENTS

The authors thank H. Falomir for many useful discussions and comments.

This work was partially supported by ANPCyT (Agencia Nacional de Promoción Científica y Tecnológica), under grant PMT-PICT 0421 and U.N.L.P. (Universidad Nacional de La Plata), Argentina.

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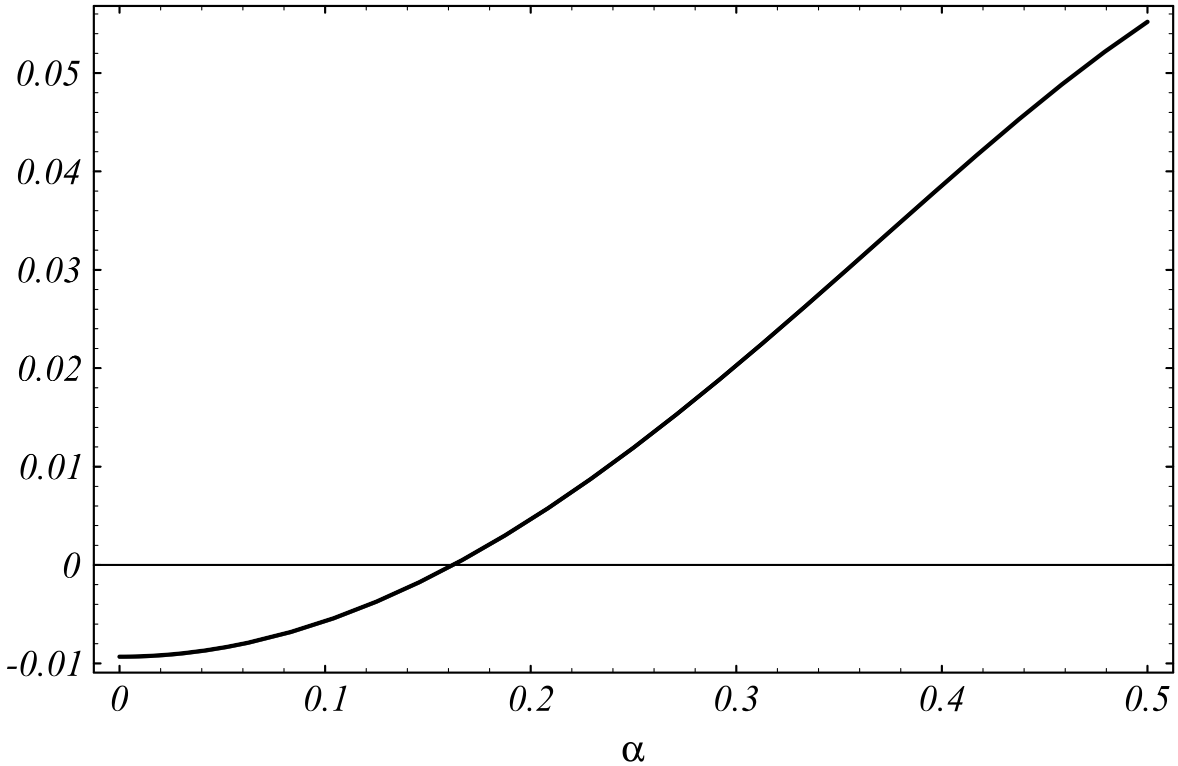

FIG. 1. Finite part of the adimensionalized Casimir Energy

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FIGURE CAPTION

FIG. 1 - Finite part of the adimensionalized Casimir Energy